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The impedance boundary condition is used to calculate the Casimir force in configurations of two parallel plates and a shpere (spherical lens) above a plate at both zero and nonzero temperature. The impedance approach allows one to find the Casimir force between the realistic test bodies regardless of the electromagnetic fluctuations inside the media. Although this approach is an approximate one, it has wider areas of application than the Lifshitz theory of the Casimir force. The general formulas of the impedance approach to the theory of the Casimir force are given and the formal substitution is found for connecting it with the Lifshitz formula. The range of micrometer separations between the test bodies which is interesting from the experimental point of view is investigated in detail. It is shown that at zero temperature the results obtained on the basis of the surface impedance method are in agreement with those obtained in framework of the Lifshitz theory within a fraction of a percent. The temperature correction to the Casimir force from the impedance method coincides with that from the Lifshitz theory up to four significant figures. The case of millimeter separations which corresponds to the normal skin effect is also considered. At zero temperature the obtained results have good agreement with the Lifshitz theory. At nonzero temperature the impedance approach is not subject to the interpretation problems peculiar to the zero-frequency term of the Lifshitz formula in dissipative media.

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I. INTRODUCTION

Currently the Casimir effect [1] has been attracted much attention owing to its promising applications in both fundamental physics and nanotechnology (see the recent review [2]). In its simplest form the Casimir effect consists in the appearance of a force between two parallel uncharged metallic plates separated by a distance a. This is a purely quantum effect caused by the alteration of the zero-point oscillations of the electromagnetic field due to the presence of the plates. Experimentally, very precise measurements of the Casimir force have been performed recently by several authors [3–9]. Theoretically, different configurations have been considered and the role of more realistic conditions, such as nonzero temperature [10–12], finite conductivity of the boundary metal [13–16] and surface roughness [17–19], carefully examined.

Fundamental applications of the Casimir effect belong to the domains of the Kaluza-Klein supergravity, quantum chromodynamics, atomic physics and condensed matter [2,20,21]. In the last few years it has been used to obtain stronger constraints on the constants of hypothetical long-range interactions [22–24]. Concerning nanotechnology the first microelectromechanical system actuated by the Casimir effect should be mentioned [9]. The nontrivial boundary dependence of the Casimir force (see, e.g., [5,25,26]) opens up new possibilities of using the Casimir effect in technology.

All the above-listed applications require precise methods of calculation of the Casimir force which take account of different influential factors and could be applied to various configurations. The most fundamental basis for the calculation of the Casimir force for realistic media is suggested by Lifshitz theory [27,28]. According to this theory the vacuum oscillations are modeled by a randomly fluctuating electromagnetic field propagating inside a dielectric material described by a frequency-dependent dielectric permittivity. Lifshitz theory gives the possibility of calculating the Casimir force taking into account both nonzero temperature and finite conductivity of the boundary metal [10–15]. It faces some problems, however, concerning the interpretation of the zero-frequency contribution to the Casimir force between metals at nonzero temperature [12].

The other method that permits to calculate the Casimir force between the realistic media is based on the use of impedance boundary conditions. This method allows one to find the Casimir force between realistic media similarly to the way it was found in Quantum Field Theory, i.e. by imposing some boundary conditions on the bounding surfaces and not considering the regions occupied by the media. The advantages of the impedance method are the following. It gives the possibility to carry out all calculations more simply. It also permits to obtain results in the

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cases Lifshitz theory has problems due to indefinite value of the zero-frequency term [12] or when the characteristic frequencies correspond to the anomalous skin effect, or in the case of low-temperature superconductors where the dielectric permittivity is not a well-defined quantity [20].

Surface impedance was first applied in Ref. [29] to calculate the Casimir force acting between two metallic plates made of real metal at zero temperature (a similar in spirit approach was used in [30] where the reflection coefficients of electromagnetic waves from metal plates were approximatelly expressed in terms of the impedance function). Recent trends increasingly make use of impedance boundary conditions in computer simulation studies of the scattering of light from metal films [31–33]. The elimination of the electromagnetic field inside the scattering medium by the use of an impedance boundary condition leads to a significant reduction of computation time with no notable loss of accuracy [31–33]. Furthermore, the use of impedance makes it possible to consider curved or rough surfaces.

In this paper we first apply impedance boundary conditions to calculate the Casimir force between real metals at nonzero temperature. Both the configurations of two plates and a sphere (spherical lens) above a plate are considered. The separations between the test bodies are examined from one tenth of a micrometer to around a hundred micrometers which cover the visible spectrum and infrared optics. The smallest separations of this interval correspond to the area of interest of modern experiments [3–9] on measuring the Casimir force. At zero temperature the impedance approach agrees with the Lifshitz formula up to a fraction of a percent. It is shown that the results for the temperature correction to the Casimir force obtained via the surface impedance are in agreement with those obtained in frames of Lifshitz theory up to four significant figures. The region of the normal skin effect is also examined, which corresponds to the space separations between test bodies larger than one millimeter. This case is interesting from a methodological point of view because in the presence of dissipation at nonzero temperature the zero-frequency term of the Lifshitz formula has an indefinite character [12]. The impedance approach was demonstrated to have no indefinite contributions. It agrees with the Lifshitz formula at zero temperature in the region of the normal skin effect and leads to quite reasonable results at nonzero temperature.

The paper is organized as follows. In Sec.II the main points of the impedance approach to the theory of the Casimir force acting between real media are outlined. Sec.III contains calculations of the Casimir force at zero temperature in the separation range corresponding to visible light and infrared optics, and the comparison of the obtained results with those found by considering Lifshitz theory. In Sec.IV the Casimir force at nonzero temperature is computed in the region of visible spectrum and infrared optics. The excellent agreement of the obtained temperature correction with the results based on Lifshitz theory is shown. Sec.V is devoted to numerical and analytical calculations of the Casimir force in the region of the normal skin effect. Both cases of zero and nonzero temperature are considered. In Sec.VI the reader will find conclusion and discussion.

II. IMPEDANCE APPROACH TO THE THEORY OF THE CASIMIR EFFECT

It is common knowledge [34] that the penetration of an electromagnetic field into a real metal can be effectively described by imposing an impedance condition on the boundary surface Γ

$$\boldsymbol{E}_t|_\Gamma = Z(\omega)\,(\boldsymbol{H}_t \times \boldsymbol{n})\,|_\Gamma. \tag{1}$$

Here $Z(\omega)$ is the impedance, $\boldsymbol{n}$ is the internal normal to the boundary surface and the lower index t denotes the tangential component of the electric and magnetic fields. If a non-magnetic medium can be characterized by some definite dielectric permittivity $\varepsilon(\omega)$, then the impedance is expressed in terms of it as

$$Z(\omega) = 1/\sqrt{\varepsilon(\omega)}. \tag{2}$$

It is known that ReZ is proportional to the time-averaged energy flux through the boundary surface. For an ideal metal ($\varepsilon = \infty$ at all frequencies) it follows that $Z(\omega) = 0$ and we return from (1) to Dirichlet boundary condition that is most often used in the field-theoretical approach to the theory of the Casimir effect.

To find the eigenfrequencies of the problem one should solve Maxwell equations with the condition (1) on the boundary surfaces. The Casimir energy can be obtained as half the sum of these eigenfrequencies, which is most simply calculated by the use of the argument theorem. By this means the result is expressed in terms of the impedance function at the imaginary frequencies. The subtraction of infinities is carried out by requiring that for the infinitely remote boundary surfaces the physical vacuum energy must be zero. The above-mentioned calculation was performed in [29] (see also [20]) at zero temperature for the case of two semispaces modelling two parallel plates separated by an empty gap $-a/2 \leq z \leq a/2$. The result for the energy per unit area is

$$E_{pp}^{(0)}(a) = \frac{\hbar}{(2\pi)^2}\int_0^\infty d\zeta \int_0^\infty Q\,dQ \ln\left(D^{\parallel} D^{\perp}\right), \tag{3}$$

where $\boldsymbol{Q}$ is a two-dimensional wave vector in the plane of plates, $Q = |\boldsymbol{Q}|$, the frequency on the imaginary axis is $\omega = i\zeta$, and the upper index (0) refers to the case of zero temperature. The explicit form of functions D for parallel and perpendicular polarizations, respectively, is given by

$$D^{\|} = D^{(0)}\left[1 + \frac{4\zeta R Z(i\zeta)}{(R+\zeta Z(i\zeta))^2}\,\frac{1}{e^{2Ra}-1}\right], \qquad D^{\perp} = D^{(0)}\left[1 + \frac{4\zeta R Z(i\zeta)}{(\zeta+RZ(i\zeta))^2}\,\frac{1}{e^{2Ra}-1}\right]. \tag{4}$$

Here the following notations are introduced

$$D^{(0)} \equiv 1 - e^{-2Ra}, \qquad R \equiv \sqrt{\frac{\zeta^2}{c^2} + Q^2}. \tag{5}$$

Note that for ideal metal $Z(i\zeta) = 0$, $D^{\|} = D^{\perp} = D^{(0)}$ and Eq.(3) immediately leads to

$$E_{pp}^{(0,0)}(a) = \frac{\hbar}{2\pi^2}\int_0^\infty d\zeta \int_0^\infty Q\,dQ \ln\left(1 - e^{-2Ra}\right) = -\frac{\pi^2\hbar c}{720a^3}, \tag{6}$$

which is indeed correct for the ideal metal [2,20,21] (the second upper index denotes the case of ideal metal).

For further computation it is convenient to introduce the dimensionless variables

$$y = 2Ra, \qquad \xi = \frac{2a\zeta}{c}. \tag{7}$$

In terms of these variables Eqs.(3) and (4) take the form

$$E_{pp}^{(0)}(a) = \frac{\hbar c}{32\pi^2 a^3}\int_0^\infty d\xi \int_\xi^\infty y\,dy \left\{2\ln\left(1-e^{-y}\right) + \ln\left[1 + \frac{X^{\|}(y,\xi)}{e^y - 1}\right] + \ln\left[1 + \frac{X^{\perp}(y,\xi)}{e^y-1}\right]\right\}, \tag{8}$$

where the quantities $X^{\|,\perp}(y,\xi)$ are given by

$$X^{\|}(y,\xi) = \frac{4\xi y Z}{(y+\xi Z)^2}, \quad X^{\perp}(y,\xi) = \frac{4\xi y Z}{(\xi + yZ)^2}, \qquad Z \equiv Z\left(i\frac{c\xi}{2a}\right). \tag{9}$$

Note that the first contribution in the right-hand side of Eq.(8) describes the case of an ideal metal (see Eq.(6)). This integral can be but will not be carried out explicitly since in the case of nonzero temperature, considered below, the representation of (8) it terms of the integrals is more appropriate.

The Casimir force per unit area of the plates is obtained from Eq.(8) as

$$F_{pp}^{(0)}(a) = -\frac{\partial E_{pp}^{(0)}(a)}{\partial a} = -\frac{\hbar c}{32\pi^2 a^4}\int_0^\infty d\xi \int_\xi^\infty y^2\,dy \left[\frac{1 - X^{\|}(y,\xi)}{e^y - 1 + X^{\|}(y,\xi)} + \frac{1 - X^{\perp}(y,\xi)}{e^y - 1 + X^{\perp}(y,\xi)}\right]. \tag{10}$$

It is notable that if we substitute the quantities $X^{\|,\perp}$, as defined in Eq.(9), for

$$X_L^{\|}(y,\xi) = \frac{4yZ\sqrt{\xi^2 + (y^2-\xi^2)Z^2}}{(y + Z\sqrt{\xi^2 + (y^2-\xi^2)Z^2})^2}, \qquad X_L^{\perp}(y,\xi) = \frac{4yZ\sqrt{\xi^2 + (y^2-\xi^2)Z^2}}{(yZ + \sqrt{\xi^2 + (y^2-\xi^2)Z^2})^2}, \tag{11}$$

and take Eq.(2) into account, then Eqs.(8), (10) would coincide with the Lifshitz result for the Casimir energy density and force. What this means is that the impedance approach ($X^{\|,\perp}$ are defined by Eq.(9)) is obtained from the Lifshitz formula if we put $y^2 - \xi^2 = 0$ in Eq.(11). Taking Eqs.(5) and (7) into account one can conclude that the impedance approximation to Lifshitz theory corresponds to the dominant contribution of the photons with a small wave vector in the plane of plates.

The results obtained for the configuration of two parallel plates can be easily adapted for the configuration of a sphere (spherical lens) above a plate. For this purpose the proximity force theorem [35] can be used. According to this theorem the force acting between a sphere and a plate is expressed via the energy density between the two parallel plates as

$$F_{ps}^{(0)}(a) = 2\pi R E_{pp}^{(0)}(a) = \frac{\hbar c R}{16\pi a^3}\int_0^\infty d\xi \int_\xi^\infty y\,dy \tag{12}$$
$$\times\left\{2\ln\left(1-e^{-y}\right) + \ln\left[1 + \frac{X^{\|}(y,\xi)}{e^y-1}\right] + \ln\left[1 + \frac{X^{\perp}(y,\xi)}{e^y-1}\right]\right\}.$$

For the configurations with small deviations from plane parallel geometry the exactness of the proximity force theorem and the other approximative methods is very high. Thus for the two plates inclined at a small angle one to another the results obtained by the use of the additive summation method differ from the exact ones by less than $10^{-2}\%$ [36]. For the configuration of a sphere above a plate the error introduced by the proximity force theorem is of order a/R [37]. To illustrate, that is of the order of 0.1% for the experimental parameters of Refs. [4–7].

All the above results were formulated for the case of zero temperature. It is not difficult, however, to modify them for the case of nonzero temperature T. In order to do so one must change the integration with respect to continuous frequency in Eq.(6), as well as in Eqs.(8), (10), (12) for the summation over the Matsubara frequencies

$$\zeta_l = 2\pi \frac{k_B T}{\hbar} l, \tag{13}$$

where $l = 0, 1, 2, \ldots$, and k_B is the Boltzmann constant. In accordance with Eq.(7) the dimensionless Matsubara frequencies are given by

$$\xi_l = 2\pi \frac{T}{T_{eff}} l, \qquad k_B T_{eff} = \frac{\hbar c}{2a}, \tag{14}$$

T_{eff} being the effective temperature. As a result, the Casimir energy density between two plates at temperature T in the impedance approach takes the form

$$E_{pp}^{(T)}(a) = E_{pp}^{(T,0)}(a) + \frac{k_B T}{8\pi a^2} \sum_{l=0}^{\infty}{}' \int_{\xi_l}^{\infty} y\,dy \left\{ \ln\left[1 + \frac{X^{\|}(y,\xi_l)}{e^y - 1}\right] + \ln\left[1 + \frac{X^{\perp}(y,\xi_l)}{e^y - 1}\right]\right\}. \tag{15}$$

Here the prime near a summation sign means that the multiple 1/2 is added to the term with $l = 0$. The quantity $E_{pp}^{(T,0)}$ is the Casimir energy density at temperature T between the plates made of ideal metal. The explicit expression for it is well known [2,38,39]:

$$\begin{aligned} E_{pp}^{(T,0)}(a) &= \frac{k_B T}{4\pi a^2} \sum_{l=0}^{\infty}{}' \int_{\xi_l}^{\infty} y\,dy \ln\left(1 - e^{-y}\right) \\ &= E_{pp}^{(0,0)}(a) \left\{ 1 + \frac{45}{\pi^3} \sum_{n=1}^{\infty} \left[\left(\frac{T}{T_{eff}}\right)^3 \frac{1}{n^3} \coth\left(\pi n \frac{T_{eff}}{T}\right) \right.\right. \\ &\quad \left.\left. + \pi \left(\frac{T}{T_{eff}}\right)^2 \frac{1}{n^2} \sinh^{-2}\left(\pi n \frac{T_{eff}}{T}\right)\right] - \left(\frac{T}{T_{eff}}\right)^4 \right\}, \end{aligned} \tag{16}$$

where the energy density between two plates made of ideal metal at zero temperature, $E_{pp}^{(0,0)}$, was defined in Eq.(6). The force acting between a sphere (spherical lens) and a plate at nonzero temperature is obtained from Eqs.(15), (16) by multiplying the right-hand sides by $2\pi R$ (compair with Eq.(12)).

In the impedance approach the force acting between two plates made of real metal at temperature T is obtained from Eq.(10) by the substitution of (14) and results in

$$F_{pp}^{(T)}(a) = -\frac{k_B T}{8\pi a^3} \sum_{l=0}^{\infty}{}' \int_{\xi_l}^{\infty} y^2\,dy \left[\frac{1 - X^{\|}(y,\xi_l)}{e^y - 1 + X^{\|}(y,\xi_l)} + \frac{1 - X^{\perp}(y,\xi_l)}{e^y - 1 + X^{\perp}(y,\xi_l)} \right]. \tag{17}$$

We conclude this section by stressing that the above method employed to get the Casimir force at nonzero temperature by changing integration for summation according to Eqs.(13), (14) is valid under certain assumptions only. Specifically, the function under the integrals in Eqs.(8), (10) must be well-defined at all points (ξ_l, y), including the point (0,0). The problems arising when this is not the case are detailly discussed in Refs. [12,40].

III. CASIMIR EFFECT IN THE REGION OF VISIBLE LIGHT AND INFRARED OPTICS AT ZERO TEMPERATURE

Here we apply the impedance approach to calculate the Casimir energy density and force when the characteristic frequencies giving the major contribution to the effect fall into the region of visible light and infrared optics. In fact, we will consider the space separations between the test bodies from one tenth of a micrometer to around a hundred micrometers. For the characteristic frequencies under consideration, as the field penetrates into the metal, it decays exponentially. The surface impedance function is pure imaginary and given by [41]

$$Z(\omega) = -\frac{i\omega}{c}\delta(\omega) = -\frac{i\omega}{\sqrt{\left(\frac{\varepsilon_F}{\hbar}\right)^2 - \omega^2}}, \tag{18}$$

where $\delta(\omega)$ is the depth of a skin layer and ε_F is the Fermi energy. It is suggested that $\omega < \varepsilon_F/\hbar$. The frequency $\varepsilon_F/\hbar$ usually belongs to the shortwave optical or near-ultraviolet parts of spectra. We consider below the case of a spherical Fermi surface when $\varepsilon_F = \hbar\omega_p$, ω_p being the effective plasma frequency. Performing the rotation to the imaginary frequency axis and using the dimensionless frequency, introduced in Eq.(7), one obtains

$$Z \equiv Z\left(\frac{ic\xi}{2a}\right) = \frac{\xi}{\sqrt{\tilde{\omega}_p^2 + \xi^2}}, \tag{19}$$

where $\tilde{\omega}_p = 2a\omega_p/c$.

If the relevant frequencies obey the inequality $\omega \ll \varepsilon_F/\hbar$ ($\xi \ll \tilde{\omega}_p$) the impedance function takes a simpler form [42]

$$Z(\omega) = -\frac{i\omega}{\omega_p}, \qquad Z = \frac{\xi}{\tilde{\omega}_p}. \tag{20}$$

In this case the skin depth is expressed as $\delta = \delta_0 \equiv c/\omega_p$ and thereby does not depend on frequency.

In Ref. [29] the approximate representation of the impedance function (20) was used to calculate perturbatively the Casimir force between two parallel plates given by Eq.(10). The result was represented as a series

$$F_{pp}^{(0)}(a) = F_{pp}^{(0,0)}(a)\sum_k c_k\left(\frac{\delta_0}{a}\right)^k, \tag{21}$$

where the relative skin depth δ_0/a is a small parameter and the Casimir force between ideal metals is $F_{pp}^{(0,0)}(a) = -\pi^2\hbar c/(240a^4)$, in accordance with Eq.(6). For the first coefficients the following values were found in [29]

$$c_0 = 1, \quad c_1 = -\frac{16}{3}, \quad c_2 = 24. \tag{22}$$

It is notable that the same coefficients were found by using Lifshitz theory [16,39], i.e. when instead of (9) the quantities (11) are substituted into Eq.(10). In the Lifshitz formula the plasma model representation for the dielectric permittivity which was used corresponds to the more exact Eq.(19) for the impedance. Expansion coefficients of higher orders were also obtained from Lifshitz theory (up to the fourth order in [16] and up to the sixth order in [43]) for both energy density and force. By way of example, in the framework of Lifshitz theory it follows [16,43] that

$$c_3^{(L)} = -\frac{640}{7}\left(1 - \frac{\pi^2}{210}\right) \approx -87.13, \qquad c_4^{(L)} = \frac{2800}{9}\left(1 - \frac{163\pi^2}{7350}\right) \approx 243.01, \tag{23}$$

whereas by considering the impedance approach, with the more exact Eq.(19), it is not difficult from Eq.(10) to arrive at

$$c_3^{(i,e)} = -\frac{640}{7}\left(1 + \frac{\pi^2}{280}\right) \approx -94.65, \qquad c_4^{(i,e)} = \frac{2800}{9}\left(1 + \frac{5\pi^2}{294}\right) \approx 363.33. \tag{24}$$

It is seen that there are differences between the numerical values of the higher order coefficients obtained from the impedance approach and those obtained from Lifshitz theory, even if one uses the more exact expression for the impedance. If we use the approximate expressions of Eq.(20) the higher order coefficients calculated in frames of the impedance approach are

$$c_3^{(i,a)} = -\frac{11520}{7\pi^4}\left[\zeta(3) + \frac{1}{8}\zeta(5)\right] \approx -22.50, \qquad c_4^{(i,a)} = \frac{14000}{3\pi^4}\left[\zeta(3) + \frac{1}{2}\zeta(5)\right] \approx 82.43, \tag{25}$$

where $\zeta(z)$ is the Riemann zeta function. That is in even worse agreement with the values of Eq.(23) obtained from Lifshitz theory. The perturbation results obtained for the energy density between the plates (i.e. for the force between a plate and a spherical lens) are in perfect analogy with the above ones.

Largely, however, the results obtained from the impedance approach and those from Lifshitz theory agree closely with each other in a wide separation range. The measure of the agreement between both approaches is represented by the quantity

$$\delta F_{pp}^{(0)}(a) = \frac{F_{pp,L}^{(0)}(a) - F_{pp}^{(0)}(a)}{F_{pp,L}^{(0)}(a)}, \tag{26}$$

where the Casimir force between the plates $F_{pp}^{(0)}(a)$ is calculated from Eq.(10) (impedance approach), and $F_{pp,L}^{(0)}(a)$ is computed in the framework of Lifshitz theory (by the use of Eq.(11)). The results calculated for *Al* ($\omega_p \approx 1.9 \times 10^{16}\,$rad/s [44]) are presented in Fig. 1 (computations were performed by the use of Mathematica).

Curve 1 corresponds to the more exact impedance function of Eq.(19), whereas curve 2 is computed by using Eq.(20). As it is seen from curve 1, for all separations from 100 nm to 10 μm the impedance approach is in good agreement with Lifshitz theory. The largest deviation (smaller than 0.5%) occurs at the smallest separation only. Regarding the approximate impedance function of Eq.(20) (curve 2), the error of impedance method exceeds 5% at separations smaller than 200 nm. However, for separations larger than 1.2 μm the relative error is less than 1%.

For the energy density between the plates (force in a configuration of a spherical lens above a plate) the impedance approach leads to even more exact results. Here the error can be characterized by

$$\delta E_{pp}^{(0)}(a) = \frac{E_{pp,L}^{(0)}(a) - E_{pp}^{(0)}(a)}{E_{pp,L}^{(0)}(a)}, \tag{27}$$

where $E_{pp}^{(0)}(a)$ is determined by Eq.(8). The results computed by the use of Mathematica are presented in Fig. 2 (curve 1 corresponds to the impedance function of Eq.(19) and curve 2 is based on the approximate Eq.(20)). As is seen from curve 1, the results for the impedance approach practically coincide with the results of Lifshitz theory in the separation range $0.4\,\mu\text{m} \leq a \leq 10\,\mu\text{m}$. In the separation range $0.1\,\mu\text{m} \leq a \leq 0.4\,\mu\text{m}$ the error introduced by the impedance approach is less than 0.3% (this maximal error occurs at the smallest separation). With the approximate impedance function of Eq.(20) the error introduced by the impedance approach is less than 1% at $a \geq 0.7\,\mu$m, and does not exceed 5% at the separations $100\,\text{nm} \leq a \leq 700\,\text{nm}$.

Thus the impedance approach given by Eqs.(8), (10), (12), and (19) seems to provide a good description of the Casimir force between metals at zero temperature with required accuracy.

IV. CASIMIR FORCE IN THE REGION OF INFRARED OPTICS AT NONZERO TEMPERATURE

In the impedance approach the Casimir energy density and force at nonzero temperature are given by Eqs.(15)–(17). If we change in these formulas $X^{\parallel,\perp}$ by $X_L^{\parallel,\perp}$, defined by Eq.(11), one obtains the Lifshitz results for the Casimir energy density and force. By applying the Poisson summation formula it is possible to rewrite Eqs.(15), (17) in the form of separated contributions of zero temperature (as it is done in [39] for the case of the Lifshitz theory)

$$E_{pp}^{(T)}(a) = E_{pp}^{(0)}(a) + \Delta_T E_{pp}(a), \qquad F_{pp}^{(T)}(a) = F_{pp}^{(0)}(a) + \Delta_T F_{pp}(a). \tag{28}$$

Here the zero-temperature contributions are defined by Eqs.(8), (10) and were computed in Sec.III. The temperature corrections in the impedance approach are given by

$$\begin{aligned} \Delta_T E_{pp}(a) = {} & \frac{\hbar c}{16\pi^2 a^3} \sum_{l=1}^{\infty} \int_0^\infty y\,dy \int_0^y d\xi \cos\left(l\xi \frac{T_{eff}}{T}\right) \Big\{ 2\ln\left(1 - e^{-y}\right) \\ & + \ln\left[1 + \frac{X^{\parallel}(y,\xi)}{e^y - 1}\right] + \ln\left[1 + \frac{X^{\perp}(y,\xi)}{e^y - 1}\right]\Big\}, \end{aligned} \tag{29}$$

$$\begin{aligned} \Delta_T F_{pp}(a) = {} & -\frac{\hbar c}{16\pi^2 a^4} \sum_{l=1}^{\infty} \int_0^\infty y^2\,dy \int_0^y d\xi \cos\left(l\xi \frac{T_{eff}}{T}\right) \\ & \times \left[\frac{1 - X^{\parallel}(y,\xi)}{e^y - 1 + X^{\parallel}(y,\xi)} + \frac{1 - X^{\perp}(y,\xi)}{e^y - 1 + X^{\perp}(y,\xi)}\right]. \end{aligned} \tag{30}$$

Let us start with the temperature correction to the energy density and expand (29) up to the second power in the small parameter δ_0/a. Note that this expansion does not depend on whether we use the more exact Eq.(19) or the approximate Eq.(20) for the impedance function. In both cases the result is

$$\begin{aligned}
\Delta_T E_{pp}(a) = & -\frac{\hbar c}{8\pi^2 a^3}\sum_{l=1}^{\infty}\left\{\frac{\pi}{2(lt)^3}\coth(\pi lt) - \frac{1}{(lt)^4} + \frac{\pi^2}{2(lt)^2}\frac{1}{\sinh^2(\pi lt)}\right. \\
& + \frac{\delta_0}{a}\left[\frac{\pi}{(lt)^3}\coth(\pi lt) - \frac{4}{(lt)^4} + \frac{\pi^2}{(lt)^2}\frac{1}{\sinh^2(\pi lt)} + \frac{2\pi^3}{lt}\frac{\coth(\pi lt)}{\sinh^2(\pi lt)}\right] \\
& - \left(\frac{\delta_0}{a}\right)^2\left[\frac{\pi}{(lt)^5} + \frac{2\pi^4}{\sinh^2(\pi lt)}\left(1 - 3\coth^2(\pi lt) + \frac{\coth(\pi lt)}{\pi lt} - \frac{1}{(\pi lt)^2}\right)\right. \\
& \left.\left. + \frac{6}{\pi(lt)^5}\left(2\pi lt\ln\left(1 - e^{-2\pi lt}\right) - \frac{2\pi^2(lt)^2}{e^{2\pi lt}-1} - \mathrm{Li}_2\left(e^{-2\pi lt}\right)\right)\right]\right\},
\end{aligned} \tag{31}$$

where $\mathrm{Li}(z)$ is the polylogarithmic function and $t \equiv T_{eff}/T$. It is remarkable that Eq.(31) coincides with the temperature correction obtained from Lifshitz theory in the same approximation [40]. Because of this, the impedance approach, when applied to calculate the temperature correction to the Casimir force, produces even more exact results than at zero temperature. In fact, the results computed by the perturbative Eq.(31) coincide with those computed numerically by the exact Lifshitz formula up to four significant figures in all separation range under consideration, from $0.1\,\mu$m to $10\,\mu$m.

Now we consider the temperature correction to the Casimir force given by Eq.(30). Expanding (30) in a power series of the small parameter δ_0/a and keeping terms up to the second order one obtains

$$\begin{aligned}
\Delta_T F_{pp}(a) = & -\frac{\hbar c}{8\pi^2 a^4}\sum_{l=1}^{\infty}\left\{\frac{1}{(lt)^4} - \frac{\pi^3}{lt}\frac{\coth(\pi lt)}{\sinh^2(\pi lt)}\right. \\
& + \frac{\delta_0}{a}\frac{\pi^3}{lt\sinh^2(\pi lt)}\left[\frac{1}{(\pi lt)^2}\sinh(\pi lt)\cosh(\pi lt) + 4\coth(\pi lt)\right. \\
& \qquad \left. + 2\pi lt - 6\pi lt\coth^2(\pi lt) + \frac{1}{\pi lt}\right] \\
& + 3\left(\frac{\delta_0}{a}\right)^2\frac{\pi^3}{lt\sinh^2(\pi lt)}\left[-4\pi lt + 5(\pi lt)^2\coth(\pi lt) + 12\pi lt\coth^2(\pi lt)\right. \\
& \qquad \left.\left. - 8(\pi lt)^2\coth^3(\pi lt) - 4\coth(\pi lt)\right]\right\}.
\end{aligned} \tag{32}$$

This result is obtained for the representation (19) as well as for representation (20) of the impedance function. It coincides with the perturbative calculation of the temperature correction to the Casimir force in Lifshitz theory. Once more, the results of numerical computations based on Lifshitz theory agree with (32) up to four sifnificant figures in the separation range from $0.1\,\mu$m to $10\,\mu$m. This means that the impedance approach is well suited for calculating the temperature correction to the Casimir force, leading to practically exact results in a wide separation range.

V. CASIMIR FORCE IN THE REGION OF THE NORMAL SKIN EFFECT

In this Section, the case of large separations between the test bodies $a > 1\,$mm is considered. For such separations the characteristic frequencies giving the crucial contribution to the Casimir force correspond to the normal skin effect. Although at so large separations the Casimir force is very small and at present cannot be detected experimentally, some special features make this case interesting from a theoretical point of view.

In the region of the normal skin effect the surface impedance is complex, namely

$$Z(\omega) = (1 - i)\sqrt{\frac{\omega}{8\pi\sigma}}, \tag{33}$$

where σ is the conductivity of the boundary metal. It can be related to the effective plasma frequency by $\sigma = \omega_p^2/(4\pi\gamma)$ (γ is the relaxation parameter). Note that in the region of the normal skin effect the characteristic frequencies of the Casimir effect are much smaller than the relaxation parameter. Because of this the dissipation processes play an important role in the region of the normal skin effect and cannot be neglected. In terms of dimensionless quantities the impedance function (33) on the imaginary axis can be represented as

$$Z \equiv Z\left(i\frac{c\xi}{2a}\right) = \sqrt{\frac{\xi}{4\pi\tilde{\sigma}}}, \tag{34}$$

where $\tilde{\sigma} = 2a\sigma/c$. The natural small parameter of the problem is

$$\sqrt{\frac{1}{4\pi\tilde{\sigma}}} = \frac{1}{\sqrt{8\pi}}\sqrt{\frac{c}{\sigma a}} = \frac{1}{\sqrt{2}}\sqrt{\frac{\delta_0}{a}\frac{\gamma}{\omega_p}} \ll 1, \tag{35}$$

where δ_0 is the penetration depth in the domain of infrared optics (see Sec.III). Taking into account that δ_0 is of the order of several tens of nanometers and γ is usually two or three orders smaller than ω_p, one can conclude that the parameter (35) is smaller than 10^{-3}.

Consider first the case of zero temperature. Substituting (34) into Eqs.(8)–(10) and expanding up to the first order in the small parameter (35) one obtains

$$E_{pp}^{(0)}(a) = E_{pp}^{(0,0)}(a)\left[1 - \frac{405\sqrt{2}}{4\pi^4}\,\zeta\left(\frac{7}{2}\right)\sqrt{\frac{c}{\sigma a}}\right], \qquad F_{pp}^{(0)}(a) = F_{pp}^{(0,0)}(a)\left[1 - \frac{945\sqrt{2}}{8\pi^4}\,\zeta\left(\frac{7}{2}\right)\sqrt{\frac{c}{\sigma a}}\right]. \tag{36}$$

Note that the coefficients near the expansion parameter in (36) are approximately equal to 1.656 and 1.932, respectively. The second of them (for the force) was first computed numerically in [29] (see also [20]). As an example, for *Al* ($\gamma \approx 9.6 \times 10^{13}$ rad/s [44]) at $a = 1$ mm the correction to unity in Eq.(36) is approximately equal to 1.6×10^{-3} (for the energy density) and 1.9×10^{-3} (for the force), i.e., much smaller than 1%. Precisely the same results are obtained by the numerical computations considering the Lifshitz formula and the Drude model representation for the dielectric permittivity. Thus at zero temperature the impedance approach and the Lifshitz formula applied in the region of the normal skin effect are in agreement.

Now consider the case of nonzero temperature. This case as described by Lifshitz theory presents difficulties because in the region of the normal skin effect dissipation processes play an important role. According to the results of Refs. [12,40] in the presence of dissipation the zeroth term of the Lifshitz formula for the Casimir force at nonzero temperature is not well-defined. The direct application of this formula leads to non-physical results. In fact, as shown in Refs. [2,12,40], the rigorous derivation of the Lifshitz formula in the framework of the Temperature Quantum Field Theory in Matsubara formulation leaves indefinite the zero-frequency contribution to the Casimir force in dissipative media. Thus a special prescription is needed to describe the Casimir effect at nonzero temperature in the region of the normal skin effect basing on the Lifshitz formula.

By contrast, surface impedance approach can be immediately applied in the case of nonzero temperature and does not need any prescription. It follows from Eqs.(9) and (34) that

$$X^{\parallel}(y,0) = X^{\perp}(y,0) = 0, \tag{37}$$

and the zero frequency contribution to the energy density (15) and force (17) are the same as for ideal metal. The other terms in Eqs.(15, (17) also can be calculated without trouble. The results for *Al* at $a = 1$ mm are as follows.

For large separations which are typical for the normal skin effect the effective temperature defined in Eq.(14) is very low (to illustrate, $T_{eff} = 1.145$ K at $a = 1$ mm). Since it is not reasonable to further increase the separation distance we consider the dependence of the Casimir energy density and force on temperature at a fixed separation. The results are found to be very close to the ones obtained for an ideal metal. At $T = 1$ K the value of $E_{pp}^{(T)}$ computed by the impedance method is equal to 0.99992 of the value computed by Eq.(16) for an ideal metal (to compare, the same quantity computed from the Lifshitz formula appended by the special prescription [12] is only 0.005% lower). For $T \geq 2$ K the impedance approach leads to practically the same values as obtained for an ideal metal. The Casimir force $F_{pp}^{(T)}$ at $T = 1$ K computed by the impedance approach is equal to 0.999745 of the force between ideal metals. Results of this kind are quite natural in the range of the normal skin effect because at so large separations any metal behaves like an ideal one.

VI. CONCLUSION AND DISCUSSION

As indicated above, the method of the surface impedance gives the possibility of calculating the Casimir energy density and force at both zero and nonzero temperature. The important advantage of this method lies in its simplicity.

As distinct from Lifshitz theory the realistic properties of the test body material are included into the boundary condition and the electromagnetic fluctuations inside the media are not considered. Surface impedance is a more general characteristic than the dielectric permittivity. As a result, the impedance approach may be employed in cases in which the Casimir force cannot be calculated from Lifshitz theory.

In the paper the main concepts of the impedance approach to the theory of the Casimir effect were outlined. The formal substitution which enables one to link the formalism of the impedance approach with the Lifshitz theory was found. The expressions for the Casimir energy density and force at nonzero temperature were obtained first in terms of the surface impedance (Sec.II). The obtained expressions were applied to calculate the Casimir force in the configuration of two parallel plates and a sphere (spherical lens) above a plate in two important cases. The first case corresponds to the space separations between the test bodies of order of one micrometer. Here the visible and infrared frequencies contribute most to the Casimir force. This case is important from an experimental point of view. The second case corresponds to separations larger than one millimeter, which is the region of the normal skin effect.

The results obtained in the impedance approach were carefully compared with those calculated using Lifshitz theory. In the separation range $0.1\,\mu\mathrm{m} \leq a \leq 10\,\mu\mathrm{m}$ at zero temperature the deviation between both approaches is less than 0.5% (for two parallel plates) and 0.3% (a sphere above a plate) when the impedance function of Eq.(19) is used. The maximal errors mentioned above occur at the smallest separation. At $a \geq 400\,\mathrm{nm}$ the results are practically coincident with those of Lifshitz theory.

The impedance approach was shown to be well suited for calculating the temperature correction to the Casimir force. In the region of the visible light and infrared optics it leads to the same perturbative results as the Lifshitz formula (see Sec.IV). The agreement between the impedance approach and the numerical computations in Lifshitz theory, at least up to four significant figures, was demonstrated.

Although at separations larger than 1 mm (the region of the normal skin effect) the Casimir force is too small to be observed experimentally, this case is interesting from a methodological point of view. At zero temperature the corrections to the Casimir energy density and force were found analytically (Eq.(36)). They are in agreement with the numerical computations performed by using the Lifshitz formula. At nonzero temperature the case of the normal skin effect presents difficulties when Lifshitz theory is used to describe the Casimir force. The reason is that in the presence of dissipation the zero-frequency term of the Lifshitz formula is not well-defined. As shown in Sec.V, the impedance approach does not present any difficulty when applied to calculate the Casimir force in the region of the normal skin effect. In Sec.V practically the same results for the Casimir force were obtained in the impedance approach as for an ideal metal. These results are approximately equal to those obtained from the Lifshitz formula appended by special prescription.

In conclusion, it may be said that the impedance approach to the theory of the Casimir effect is rather effective and provides the means for precise calculations of the Casimir force acting between realistic media in a wide separation range.

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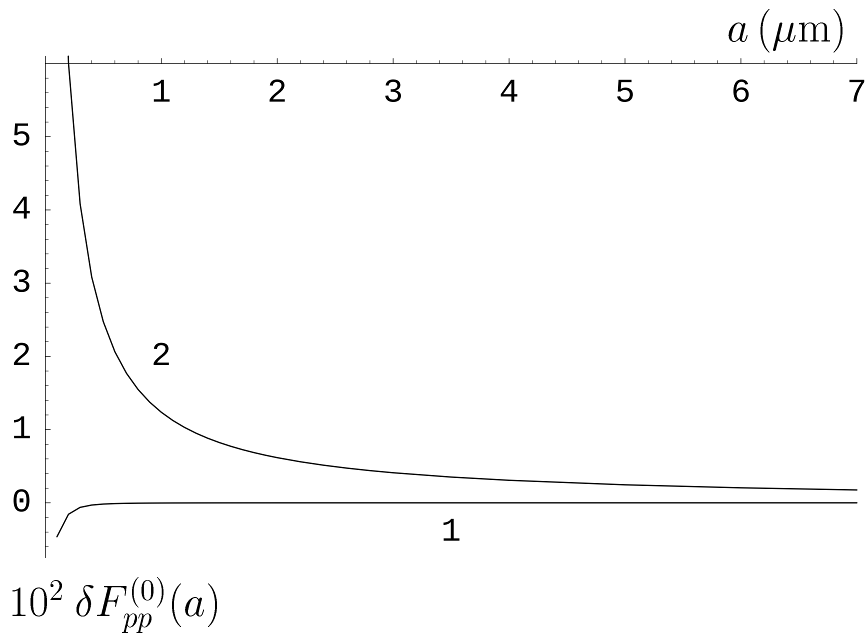

FIG. 1. The relative deviation of the Casimir force between parallel plates at zero temperature as computed in the impedance approach and from Lifshitz theory. Curve 1 corresponds to the impedance function of Eq.(19) and curve 2 to the impedance of Eq.(20).

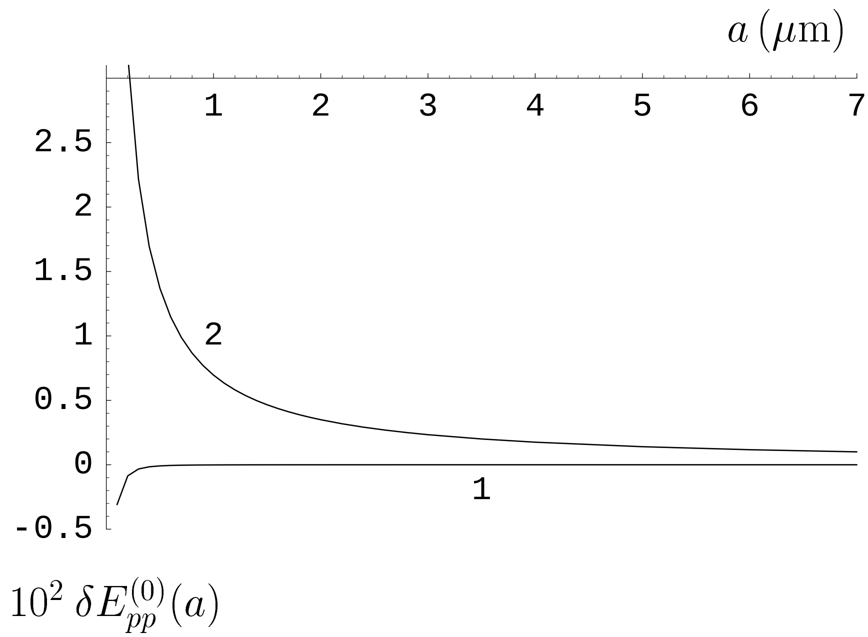

FIG. 2. The relative deviation of the Casimir energy density between parallel plates at zero temperature as computed in the impedance approach and from Lifshitz theory. Curve 1 corresponds to the impedance function of Eq.(19) and curve 2 to the impedance of Eq.(20).